Exam 3 review

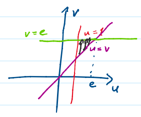
Thursday, November 21, 2024 5:01 PM

Change of variables:

9) $\iint_R \frac{y}{x-3y}\, \partial A$ R: $y=1$, $y=\frac{1}{4}x$, $x-3y=e$ Use $x=3u+v$, $y=u$

Ans: Region R':
- $u=1$
- $u=\frac{1}{4}(3u+v) \leadsto \frac{1}{4}u=\frac{1}{4}v \Rightarrow u=v$
- $3u+v-3u=e \Rightarrow v=e$

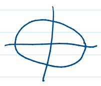
(1) $u=1$
(2) $u=v$
(3) $v=e$



Jacobian: $J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \leadsto \det(J)=-1$

$$\iint_R \frac{y}{x-3y}\,\partial A = \iint_{R'} \frac{u}{3u+v-3u} \cdot |\det(J)| = \int_{u=1}^{u=e}\int_{v=u}^{v=e} \frac{u}{v}\, \partial v\, \partial u = \text{computations}$$

- Find area inside $\frac{x^2}{25}+\frac{y^2}{36}=1$, with $\frac{x^2}{25} = r^2\cos^2(\theta)$ and $\frac{y^2}{36} = r^2\sin^2(\theta)$



$x^2+y^2=1$ $x=r\cos(\theta)$ $y=r\sin(\theta)$

$r^2\cos^2(\theta)+r^2\sin^2(\theta)=r^2$

$\frac{x^2}{25}=r^2\cos^2(\theta) \Rightarrow \boxed{x=5r\cos(\theta)}$

$\frac{y^2}{36}=r^2\sin^2(\theta) \rightarrow \boxed{y=6r\sin(\theta)}$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 5\cos(\theta) & -5r\sin(\theta) \\ 6\sin(\theta) & 6r\cos(\theta) \end{vmatrix} \leadsto \det(J)=30r(\cos^2(\theta)+\sin^2(\theta))=30r$$

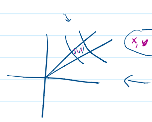
Region: $\frac{x^2}{25}+\frac{y^2}{36}=1 \leadsto r^2\cos^2(\theta)+r^2\sin^2(\theta)=1 \Rightarrow \boxed{r=1}$

Integral: $\int_0^{2\pi}\int_0^1 1\cdot|\det(J)| \cdot \partial r\, \partial\theta = \int_0^{2\pi}\int_0^1 30r\, \partial r\, \partial\theta$

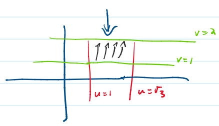
2 Problem set R: $y=x \leadsto \frac{y}{x}=1$, $y=\sqrt{3}x \leadsto \frac{y}{x}=\sqrt{3}$, $xy=1$, $xy=2$ Compute $\iint_R y^2\, \partial A$

$v=xy$

$u=\frac{y}{x}$
$v=xy$



New Region:
(1) $u=1$
(2) $u=\sqrt{3}$
(3) $v=1$
(4) $v=2$



$\left.\begin{array}{l} u=\frac{y}{x} \\ v=xy \end{array}\right\} \Rightarrow uv=y^2 \Rightarrow \boxed{y=\sqrt{uv}}$

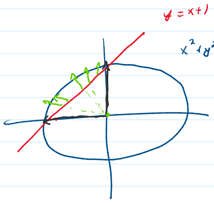
$\left.\begin{array}{l} u=\frac{y}{x} \\ v=xy \end{array}\right\} \frac{v}{u}=x^2 \rightarrow \boxed{x=\sqrt{\frac{v}{u}}}$

$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \text{computations...}$

$\int_{u=1}^{u=\sqrt{3}}\int_{v=1}^{v=2} \cancel{x}\, uv \cdot |\det(J)|\, \partial v\, \partial u = \text{computations}$

If you have to find your own change of variables
- ellipse → circle
- obvious like before

Polar:



$$A=\int_{\theta=\frac{\pi}{2}}^{\theta=\pi}\int_{r=\text{line}}^{r=\text{circle}} r\, \partial r\, \partial\theta$$

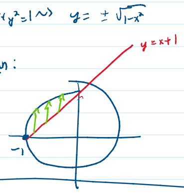
$r=1$, $r=\frac{1}{\sin(\theta)-\cos(\theta)}$

$y=x+1$
$r\sin(\theta)=r\cos(\theta)+1$
$r(\sin(\theta)-\cos(\theta))=1$
$r=\frac{1}{\sin(\theta)-\cos(\theta)}$

$x^2+y^2=1 \leadsto y=\pm\sqrt{1-x^2}$

$$=\int_{\pi/2}^{\pi}\int_{\frac{1}{\sin(\theta)-\cos(\theta)}}^{1} r\, \partial r\, \partial\theta$$

Cartesian:

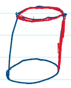


$$\int_{x=-1}^{x=0}\int_{y=\text{line}}^{y=\text{upper circle}} 1\, \partial y\, \partial x = \int_{x=-1}^{x=0}\int_{y=x+1}^{y=\sqrt{1-x^2}} 1\, \partial y\, \partial x$$

Parametrization:

Cone $\qquad\qquad \rho, \theta, \varphi$

1) Cylinder: $x^2+y^2=4 \rightsquigarrow \boxed{r=2}$



$$r(\theta, z) = \begin{bmatrix} 2\cos(\theta) \\ 2\sin(\theta) \\ z \end{bmatrix}$$

2) $z^2 = x^2+y^2 \rightsquigarrow \boxed{\varphi = \frac{\pi}{4}}$

$$r(\rho, \theta) = \begin{bmatrix} \rho\cos(\theta)\sin(\frac{\pi}{4}) \\ \rho\sin(\theta)\sin(\frac{\pi}{4}) \\ \rho\cos(\frac{\pi}{4}) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} \rho\cos(\theta) \\ \rho\sin(\theta) \\ \rho \end{bmatrix}$$

3) Plane: $2x+3y+2z=4 \rightsquigarrow \boxed{z = 2 - x - \frac{3}{2}y}$ $\qquad z = f(x,y)$

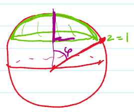
$$r(x,y) = \begin{bmatrix} x \\ y \\ 2 - x - \frac{3}{2}y \end{bmatrix} \qquad r(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

Questions: Find tangent plane

surface area

3) Σ: $x^2+y^2+z^2=4$ above $z=1$ Find area of Σ

$\to \boxed{\rho = 2}$



1st way: Actually compute r_θ, r_φ, $r_\theta \times r_\varphi$, $\|r_\theta \times r_\varphi\|$

2nd way: $\|r_\theta \times r_\varphi\| = R^2 \sin(\varphi)$

$$r(\theta, \varphi) = \begin{bmatrix} 2\cos(\theta)\sin(\varphi) \\ 2\sin(\theta)\sin(\varphi) \\ 2\cos(\varphi) \end{bmatrix}$$

$$S = \iint_R \|r_\theta \times r_\varphi\| = \iint_R 4\sin(\varphi) = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{3}} 4\sin(\varphi)\, d\varphi\, d\theta$$

$$\left.\begin{matrix} x^2+y^2+z^2=4 \\ z=1 \end{matrix}\right\} \Rightarrow x^2+y^2=3 \rightsquigarrow \rho^2\sin^2(\varphi)=3 \Rightarrow 4\sin^2(\varphi)=3 \Rightarrow \sin(\varphi)=\frac{\sqrt{3}}{2} \Rightarrow \varphi=\frac{\pi}{3}$$

$$z=1 \rightsquigarrow \rho\cos(\varphi)=1 \to 2\cos(\varphi)=1 \Rightarrow \cos(\varphi)=\frac{1}{2} \Rightarrow \varphi=\frac{\pi}{3}$$

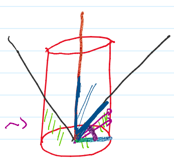
$x^2+y^2+z^2=\rho^2$

$x^2+y^2=\rho^2\sin^2(\varphi)$

(h) Volume of solid region bounded above by $z=\sqrt{x^2+y^2} \rightsquigarrow \varphi=\frac{\pi}{4}$

sides by $x^2+y^2=4 \rightsquigarrow \rho^2\sin^2(\varphi)=4 \Rightarrow \rho=\frac{2}{\sin(\varphi)}$

below by xy-plane



$$V = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\text{cone}}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=\text{cylinder}} 1\cdot\rho^2\sin(\theta) = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=\frac{2}{\sin(\varphi)}} \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

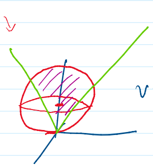
(g) $\iiint x^2+y^2\, dV$ $\quad$ D: bounded above by $x^2+y^2+z^2=4z \rightsquigarrow x^2+y^2+(z-2)^2=4$

below by $z^2=x^2+y^2$

$x^2+y^2+z^2=4z$

$\rho^2=4\rho\cos(\varphi)$

$\rho=4\cos(\varphi)$



$$V = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{4}}\int_{\rho=0}^{\rho=\text{sphere}=4\cos(\varphi)} \underbrace{\rho^2\sin^2(\varphi)}_{x^2+y^2}\ \underbrace{\rho^2\sin(\varphi)}_{\text{spherical}}\, d\rho\, d\varphi\, d\theta$$

$\rho=2a\cos(\varphi)$

$\rho^2=2a\rho\cos(\varphi)$

Quiz 9: $x^2+y^2+z^2=2az \rightsquigarrow x^2+y^2+(z-a)^2=a^2$

$x^2+y^2+z^2=az \rightsquigarrow x^2+y^2+(z-\frac{a}{2})^2=(\frac{a}{2})^2$

$\rho^2=a\rho\cos(\varphi)$

$\rho=a\cos(\varphi)$

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\rho=\text{inner sphere}=a\cos(\varphi)}^{\rho=\text{outer sphere}=2a\cos(\varphi)} \rho^2\sin^2(\varphi)\, d\rho\, d\varphi\, d\theta$$

Ch 15: $\rightsquigarrow \int_C f \cdot ds = \int_a^b f(r(t)) \, \|\frac{\partial r}{\partial t}\| \, dt$

$\rightsquigarrow \int_C F \cdot dr = \begin{cases} \int_a^b F(r(t)) \cdot \frac{\partial r}{\partial t} \, dt \\ f(\text{end pt}) - f(\text{start pt}) \quad \text{if} \quad F = \nabla f \end{cases}$

$\rightsquigarrow$ FTLI

$(\text{curl}(F) = 0)$ $\rightarrow M_y = N_x$ if $F = \begin{bmatrix} M \\ N \end{bmatrix}$

$\begin{bmatrix} N_z = P_y \\ M_z = P_x \\ M_y = N_x \end{bmatrix}$ if $F = \begin{bmatrix} M \\ N \\ P \end{bmatrix}$

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1) $F = \begin{bmatrix} x \\ 0 \\ -y^2 \end{bmatrix} = \begin{bmatrix} M \\ N \\ P \end{bmatrix}$ (a) Evaluate $\text{curl}(F)$, $\text{div}(F)$

$\text{div}(F) = M_x + N_y + P_z$

(b) Let C = intersection of $x^2+y^2=1$ and $z=1$ counterclockwise
Evaluate $\int_C F \cdot dr$

$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & -y^2 \end{vmatrix} = \begin{bmatrix} \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z} \cdot 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2y \\ 0 \\ 0 \end{bmatrix} \neq 0$

not conservative

$F = \begin{bmatrix} xy + z \\ x^2 + z^2 \\ x + y + z \end{bmatrix}$ $\quad \text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy+z & x^2+z^2 & x+y+z \end{vmatrix} = \begin{bmatrix} \frac{\partial}{\partial y}(x+y+z) - \frac{\partial}{\partial z}(x^2+z^2) \\ -\frac{\partial}{\partial x}(x+y+z) + \frac{\partial}{\partial z}(xy+z) \\ \frac{\partial}{\partial x}(x^2+z^2) - \frac{\partial}{\partial y}(xy+z) \end{bmatrix} = \begin{bmatrix} 1 - 2z \\ -1 + 1 \\ 2x - x \end{bmatrix}$

$\text{div}(F) = M_x + N_y + P_z = 1 + 0 + 0 = 1$

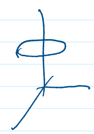
(b)

$z=1$

$x^2+y^2=1$

$r=1$

$r(t) = \begin{bmatrix} 1\cos(t) \\ 1\sin(t) \\ z=1 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1 \end{bmatrix}$ counterclockwise



$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot \frac{\partial r}{\partial t} = \int_0^{2\pi} \underbrace{\begin{bmatrix} \cos(t) \\ 0 \\ -\sin^2(t) \end{bmatrix}}_{F(r(t))} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix} dt = \int_0^{2\pi} -\cos(t)\sin(t) \, dt = 0$

$F = \begin{bmatrix} x+y \\ y^2+z+x \\ 2x+y+3z \end{bmatrix}$

$F(r(t)) = \begin{bmatrix} 2t + 3t \\ (3t)^2 + 4t + 2t \\ 2 \cdot 2t + 3t + 3 \cdot (4t) \end{bmatrix}$

$r(t) = \begin{bmatrix} 2t \\ 3t \\ 4t \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$

$\int_C (y + 2xe^y) \, dx + (x + x^2 e^y) \, dy$ $\quad C: \ r(t) = \begin{bmatrix} t-1 \\ e^{t^4} \\ t^2+1 \end{bmatrix} \quad 0 \le t \le 1$

end pt: $r(1) = \begin{bmatrix} 0 \\ e \\ 2 \end{bmatrix}$

start pt: $r(0) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$= \int_C F \cdot dr \quad F = \begin{bmatrix} y + 2xe^y \\ x + x^2 e^y \end{bmatrix}$ $\quad M_y = 1 + 2xe^y = N_x = 1 + 2xe^y$

so it is conservative

Goal: Find f s.t. $F = \nabla f$

$F = \begin{bmatrix} y + 2xe^y \\ x + x^2 e^y \end{bmatrix} = \nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$

(1) $f_x = y + 2xe^y$

$\rightsquigarrow \boxed{f(x,y) = xy + x^2 e^y + h(y)}$ $\quad f_y = x + x^2 e^y + h'(y)$

(2) $f_y = \cancel{x + x^2 e^y} = \cancel{x + x^2 e^y} + h'(y)$

$\Rightarrow h'(y) = 0$

$\Rightarrow h(y) = \text{const.}$

$\nabla f = \begin{bmatrix} y + 2xe^y \\ x + x^2 e^y \end{bmatrix}$ ✓ $\quad \boxed{f(x,y) = xy + x^2 e^y}$

So $\int_C F \cdot dr = f(\text{end pt}) - f(\text{start pt}) = f((0, e, 2)) - f((-1, 1, 1))$

$= 0 - ((-1) \cdot 1 + (-1)^2 \cdot e) = 1 - e$

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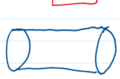
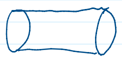
3) $S: x^2 + z^2 = 4$ between $y = 1$ and $y = 4$. Find an equation of the tangent plane at $(2, 3, 0)$

$\boxed{r = 2}$



$r(\theta, y) = \begin{bmatrix} 2\cos(\theta) \\ y \\ 2\sin(\theta) \end{bmatrix}$ $\quad \theta \in [0, 2\pi]$, $y \in [1, 4]$ $\quad r_\theta = \begin{bmatrix} -2\sin(\theta) \\ 0 \\ 2\cos(\theta) \end{bmatrix}$



$$r(\theta, y) = \begin{bmatrix} 2\cos(\theta) \\ y \\ 2\sin(\theta) \end{bmatrix}$$

$\theta \in [0, 2\pi]$
$y \in [1, 4]$

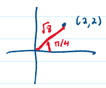
$$r_\theta = \begin{bmatrix} -2\sin(\theta) \\ 0 \\ 2\cos(\theta) \end{bmatrix}$$

$$r_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N = r_\theta \times r_y = \begin{vmatrix} i & j & k \\ -2\sin(\theta) & 0 & 2\cos(\theta) \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -2\cos(\theta) \\ 0 \\ -2\sin(\theta) \end{bmatrix} \rightsquigarrow$$

Plug in the point $P = (2, 2, 0)$

$$N(P) = \begin{bmatrix} -2\cos(\frac{\pi}{4}) \\ 0 \\ -2\sin(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix}$$



$\theta = \frac{\pi}{4}$

$r = \sqrt{8}$